ELEN 4810 Final Exam

Monday, December 19, 2022, 1:10-4:00 PM. Two sheets of handwritten notes are allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper.

There are a total of 4 questions. Good luck!

Name:

Uni:

1. Discrete Fourier Transform and Fast Fourier Transform. Consider two discrete time signals $u[n]$ and $v[n]$, which satisfy

$$\begin{array}{lll} u[n] \neq 0, & n = 0, 4, 8, 12, \ldots, 60, & u[n] = 0 \text{ else}, \\ v[n] \neq 0, & n = 0, 8, 16, 24, \ldots, 56, & v[n] = 0 \text{ else}. \end{array}$$

Set $y = u * v$. Please answer the following questions:

Part A. Suppose we compute y via the Discrete Fourier Transform, via

$$\mathrm{DFT}_N^{-1}\left(\mathrm{DFT}_N(u)[k]\; \mathrm{DFT}_N(v)[k] \right).$$

For what choices of N does this operation correctly compute y?

Part B. In this part, we use the structure of u and v to compute $y[n]$ more efficiently (similar to the Fast Fourier Transform). Let $\bar{u}$ and $\bar{v}$ be downsampled versions of u and v:

$$\begin{aligned} \bar{u} &= u \downarrow 4, \\ \bar{v} &= v \downarrow 8. \end{aligned}$$

Let

$$\begin{aligned} \bar{U}[k] &= \mathrm{DFT}_{32}\left\{ \bar{u} \right\}[k], \\ \bar{V}[k] &= \mathrm{DFT}_{16}\left\{ \bar{v} \right\}[k], \\ Y[k] &= \mathrm{DFT}_{128}\left\{ y \right\}[k]. \end{aligned}$$

Please give an expression for $Y[k]$ in terms of $\bar{U}$ and $\bar{V}$.

Answer to Problem 1: out of 6 points, 3 for each part

Part A. The nonzero entries of the signal y are located in the range $0 \leq n \leq 116$. Any $N \geq 117$ will correctly compute y.

Part B. We use the following relationship, which we developed for the decmiation-in-time-FFT: for a signal $x[n]$ of even length, we have

$$\mathrm{DFT}_N\{x\}[k] = \mathrm{DFT}_{N/2}\{x_e\}\Big[k \mod N/2\Big] \;+\; e^{-j2\pi k/N}\mathrm{DFT}_{N/2}\{x_o\}\Big[k \mod N/2\Big],$$

where

$$\begin{aligned} x_e[n] &= x[2n] \\ x_o[n] &= x[2n+1]. \end{aligned}$$

We know that

$$Y[k] \quad = \quad \mathrm{DFT}_{128}\{u\}[k]\mathrm{DFT}_{128}\{v\}[k]. \tag{1}$$

Applying the above FFT formula twice to u (noting that both times, the odd term is zero) and applying it three times to v (noting that all three times, the odd term is zero), we obtain

$$Y[k] = \bar{U}[k \mod 32]\bar{V}[k \mod 16]. \tag{2}$$

Note: this solution can also be obtained by direct calculation:

$$U[k] \quad = \quad \sum_n u[n]e^{-j2\pi kn/128} = \sum_\ell u[4\ell]e^{-j2\pi 4\ell/128} \tag{3}$$

2. Z **Transform.** Consider the following rational transfer function $H(z)$:

$$H(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}.$$

Part a. What are the poles and zeros of H?

Part b. Assuming the system is *causal*, please specify the region of convergence (ROC) and the impulse response $h[n]$.

Part c. Assuming the system is *stable*, please specify the region of convergence (ROC) and the impulse response $h[n]$.

Part d. Which of the following best describes the system?

LOW PASS BAND PASS HIGH PASS ALL PASS

Answer to Problem 2: Graded out of 8 points (2 points per part).

Part a. Poles $\rho = 1/2, 2$. Zeros $\zeta = -1,\, 0$.

Part b. Performing partial fraction expansion, we have

$$H(z) = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - 2z^{-1}},$$

with

$$A_1 = \left[\frac{1 + z^{-1}}{1 - 2z^{-1}}\right]\Bigg|_{z=\frac{1}{2}} = \frac{3}{-3} = -1 \tag{5}$$

$$A_2 = \left[\frac{1 + z^{-1}}{1 - \frac{1}{2}z^{-1}}\right]\Bigg|_{z=2} = \frac{3/2}{3/4} = 2. \tag{6}$$

If the system is causal the ROC extends outward from the largest magnitude pole:

$$\text{ROC}\{\text{h}\} = \{z \mid |z| > 2\},$$

and

$$h[n] = -\left(\frac{1}{2}\right)^n u[n] + 2 \times 2^n u[n].$$

Part c. If the system is stable, the ROC contains the unit circle, and so

$$\text{ROC}\{h\} = \{z \mid \tfrac{1}{2} < |z| < 2\}.$$

The impulse response is

$$h[n] = -\left(\frac{1}{2}\right)^n u[n] - 2 \times 2^n u[-n-1],$$

i.e., pole $z = 1/2$ corresponds to a right sided exponential sequence, and $z = 2$ corresponds to a left sided exponential sequence.

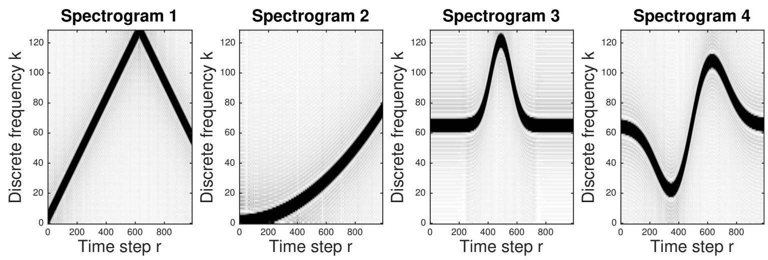
Part d. LOWPASS. The system has poles at $z = 1/2$ and $z = 2$, which correspond to angle/frequency $\omega = 0$ and a zero at $z = -1$, which corresponds to angle/frequency $\omega = \pm\pi$. The magnitude response is maximized at $\omega = 0$ and minimized at $\omega = \pm pi$.

3. Spectrograms. The following question has two parts.

Part (a). A signal $x[n]$ has the form

$$x[n] = \sin\left(\alpha n + \beta \exp\left(-\gamma(n-\tau)^2\right)\right),$$

for some scalars α, β, γ, τ.

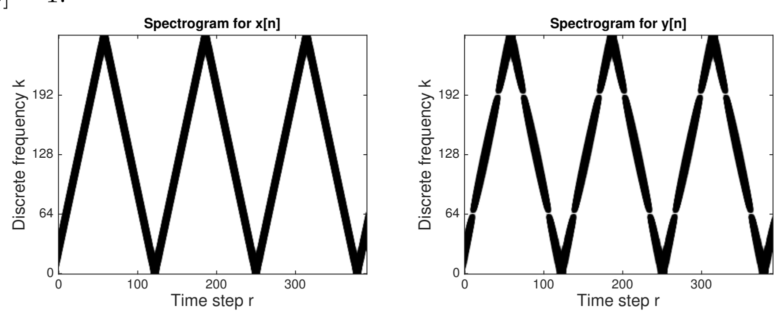


Which of the four figures above is the spectrogram of the signal? **For full credit, please justify your answer.**

Part (b). A linear chirp signal

$$x[n] = \cos\left(\alpha n^2 + \beta\right)$$

is passed through a canonical generalized linear phase system whose impulse response has length 5, and satisfies $h[0] = 1$.



Above are the spectrograms for $x[n]$ (left) and $y[n] = h * x[n]$ (right). Both spectrograms are generated with a Discrete Fourier Transform (DFT) of length $N = 512$.

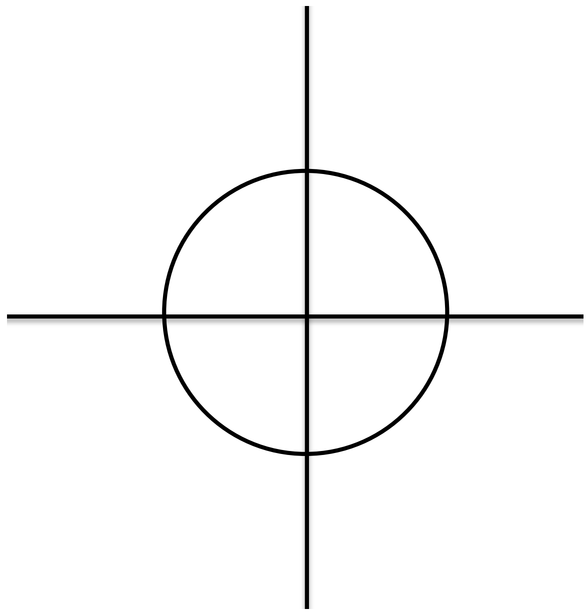
Please answer the following questions as accurately as possible, given the available information:

(b.i) What type of canonical generalized linear phase system is this?

(b.ii) What is the group delay $\text{grd}[H(e^{j\omega})]$?

(b.iii) Please sketch the pole-zero diagram of $H(z)$, using the axes on the next page. Please label any repeated poles and zeros with their multiplicity.

Answer to Problem 3: 10 Points: 4 points for part a, 2 pointes each for parts (b.i)-(b.iii).



Axis for Part b.iii

Part (a) We have $x[n] = \sin(f(n))$, with

$$\begin{aligned} f(n) &= \alpha n + \beta \exp(-\gamma(n-\tau)^2) \\ &\approx f(n_0) + \dot{f}(n_0)(n - n_0). \end{aligned}$$

Notice that

$$\dot{f}(n_0) = \alpha - 2\beta\gamma(n_0 - \tau)\exp(-\gamma(n_0-\tau)^2).$$

The instantaneous frequency at time n_0 is $\dot{f}(n_0)$; this corresponds to Spectrogram 4.

Part (b.i) TYPE I – can be deduced from the length and the fact that there are no zeros at frequency 0 or π.

Part (b.ii) $\text{grd}[H(e^{j\omega})] = 2$

Part (b.iii) The plot should exhibit zeros (of multiplicity one) at $z = e^{j\pi/4}, e^{j3\pi/4}, e^{-j\pi/4}, e^{-j3\pi/4}$.

4. Filter Design by Windowing. In this problem, we design a low-pass filter by windowing. We set

$$H_{\text{target}}(e^{j\omega}) = \begin{cases} e^{-j\omega(L-1)/2} & |\omega| \leq \frac{\pi}{4} \\ 0 & \frac{\pi}{4} < |\omega| \leq \pi. \end{cases}$$

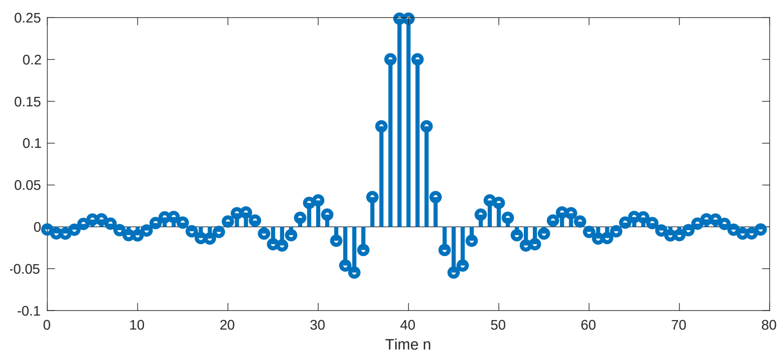
The corresponding time-domain target is

$$h_{\text{target}}[n] = \frac{\sin\left(\frac{\pi}{4}\left(n - \frac{L-1}{2}\right)\right)}{\pi\left(n - \frac{L-1}{2}\right)}$$

We use a rectangular window

$$w[n] = \begin{cases} 1 & 0 \leq n \leq L-1 \\ 0 & \text{else} \end{cases}$$

and set $h[n] = w[n]\, h_{\text{target}}[n]$. The impulse response $h[n]$ is plotted below, for $L = 80$:



Part A. Does the filter $h[n]$ have generalized linear phase? Why or why not?

Part B. In lecture, we discussed Kaiser windowing, which uses a different choice of $w[n]$. What is the main advantage of Kaiser windowing compared to the rectangular window used in part A?

Part C. What are the two main advantages of design by L^∞ optimization, compared to design by windowing?

Part D. Let $h[n]$ be our designed impulse response, $H(z)$ its Z-transform, and let $\zeta_1, \ldots, \zeta_M$ denote the zeros of $H(z)$.

Suppose we generate a new filter by setting $h_{\text{new}}[n] = (-1)^n h[n]$. **Please give an expression for the zeros $\zeta'_1, \ldots, \zeta'_M$ of $H_{\text{new}}(z)$ in terms of the zeros $\zeta_1, \ldots, \zeta_M$.**

Part E. Which of the following best characterizes the filter $h_{\text{new}}[n]$? Why?

LOW PASS BAND PASS HIGH PASS ALL PASS

4. Answer to Problem 4: Out of 11 points (part A = 2, B = 2, C = 2, D = 3, E = 2).

Part A. Yes. The impulse response is supported from 0 to 79, and is symmetric about 79/2.

Part B. Kaiser windowing selects the window shape based on the desired approximation error (tolerance δ) and hence allows us to achieve a prespecified approximation error. In contrast, the rectangular window exhibits a fixed (large) sidelobe height, which cannot be varied.

Also acceptable: Answers noting that the sidelobes of the rectangular window are large (Gibbs phenomenon) and that Kaiser windows with appropriately chosen parameter have smaller sidelobes.

Part C. Advantage 1: Design by optimization (L^∞ / Parks-McClellan) allows different tolerances δ_p, δ_s in the passband and stopband. Advantage 2: Design by optimization minimizes the approximation error for a given filter length, and hence yields a better tradeoff between approximation error and filter length, compared to windowing.

Part D. Note that

$$\begin{aligned} H_{\text{new}}(z) &= \sum_n (-1)^n h[n] z^{-n} \\ &= \sum_n h[n](-z)^{-n} \\ &= H(-z). \end{aligned}$$

So, ζ' is a zero of $H_{\text{new}}(z)$ if and only if $\zeta' = -\zeta$ for some zero of $H(z)$:

$$\zeta'_1, \ldots, \zeta'_M = -\zeta_1, \ldots, -\zeta_M.$$

Part E. HIGH PASS

$H(z)$ is a lowpass filter, and $H_{\text{new}}(e^{j\omega}) = H(-e^{j\omega}) = H(e^{j(\omega-\pi)})$. This transformation maps low-frequencies to high frequencies, and vice versa. This answer can also be explained in terms of the zeros: if H has a zero ζ at angle/frequency ω, H_{new} has a zero ζ' at angle/frequency $\omega - \pi$.